# Everything you need to know about Taylor series

In this paper, I’ll cover power series, Taylor polynomials, Taylor Series, and some application of Taylor series. This document assumes you know what a series is, and how to check if a series converges or diverges using ratio and non-ratio tests (BCT, LCT, p-test, ratio test, root test, integral test, etc.).

## Power series

A power series is a series centered at is of the form

Which I will call the standard power series (S.P.S)

There’s also the corresponding function,

Which I’ll label standard Taylor series S.T.S

Where is a sequence of real numbers. The value of will be the main driver on whether a series converges or diverges. The values of x which converge is called the radius of convergence.

***Proposition:*** Consider the SPS. There exists a unique number such that the power series converges absolutely for all and diverges for all . This R is called the *Radius of convergence*.

***Proof:*** (Holden 776)

Note that the radius of convergence does not include the end points (hence the ). Those must be checked manually. The radius of convergence plus the end points is called the interval (or domain) of convergence.

The reason it’s called the radius of convergence is because in the complex plane, the values that converge will be a circle with a radius, hence having a ‘radius of convergence’. When trying to find the radius of convergence, you’ll use the same strategy as finding the convergence of any other series, except we’ll let x vary. For example:

Find the radius and domain of convergence for the power series

Solution: This is a perfect example of the ratio test:

By the ratio test, the ratio must be less than one for it to converge, therefore:

By the ratio test, the ratio must be greater than one for it to diverge, therefore:

Which indicates that the radius of convergence is 2 (R = 2). To verify the domain of convergence, check the end points:

Which converges by p-test

Which converges by the alternating series test. Therefore, the domain of convergence is .

As mentioned earlier, is the pivot point on whether a power series converge or not. I’ll prove this here

***Proposition:*** Consider the SPS. If all but finitely many of the are non-zero, and the limit

Then the radius of convergence for the series is precisely R.

***Proof***: The ‘finitely many’ part is a technicality to let us use the ratio test. notice that this kind of like a ‘reverse ratio test’. We’ll start by the ratio test and move.

By the limit laws:

Therefore, by the ratio test, for a series to converge:

And to diverge,

Showing that R is the radius of convergence. Q.E.D.

***Example in use:***

Determine the radius and domain of convergence for the power-series

***Solution:*** let . Applying the previous the previous proposition:

This shows that the radius of convergence is , therefore

To determine the domain of convergence, plug in the end points

Which diverges by p-test

Which converges by alternative series test. Therefore, the domain of convergence is

This same could be done for the root test:

***Proposition:*** Consider the SPS. If

Then the radius of convergence is . if . If .

***Proof:*** left as an exercise

***Example:*** determine the RoC and DoC for the series

***solution***: Using the root test for power series:

therefore, .

## Power series as a function

Power series can represent a function. If SPS has a radius of convergence R, then

The proof of this requires uniform continuity and MAT157 material.

Having established the function is being continuous, we could explore the differentiation and integrability of these functions. Surprisingly, it is very simple:

***Proposition:***

Which both have Radii R as well.

***Proof***: This proof is left for MAT157.

Note that the DoC could change after differentiating or integrating could be different. For proof, try:

### Example of power series using the geometric series

It is very hard to associate a power series with a definite formula, like

But there are a few that you should know. One in particular you’ve already proven in EYTNKA sequence and series – the geometric series:

Since integration and differentiation is done term by term, we can deduce many other series using the geometric series:

***Example:***

Determine a power series representation for the function g(x) = on .

***Solution:*** take the geometric series as f(x). substitute :

You can check the the RoC is still 1. Integrating this series we get

For some reason, we need to find the constant. Just plug in a good value, like 0. You’ll end up C = 0.

***Example:***

Determine a power series representation for the function on .

***Solution:*** take f(x) to represent the geometric series. Therefore, is

We could also multiply this function by .

***Example:***

Show that he is alternating harmonic series converse to .

***Solution:***

Let f(x) be the geometric series, then

This will come in handy, since we could integrate the left and right-hand side:

We could shift the value by one without changing the value. This is very useful in this scenario. Plugging in 1, we get

## Taylor polynomials

We’ve covered what it means to relate a power series to a function, but what if we want to go the other way around – relate a function to a power series, or more precisely a polynomial. With a nice enough function, this could be done, and such a polynomial is called a Taylor polynomial. Taking an infinite polynomial of this kind is called a Taylor series.

Definition: IF f is n-times continuously differentiable at x=a, the nth order *Taylor polynomial*of *f at a is*

You should be familiar with identifying a couple of Taylor polynomials, namely the following:

|  |  |  |
| --- | --- | --- |
|  | Taylor Polynomial | Degree |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

The questions surrounded are quite easy:

***Example***: For a general , determine the *n*th order tailor polynomial of he function

***Solution***:

We need to find the value at x = 0, therefore

Therefore,

***Example:*** Find the third order Taylor polynomial for

***Solution:*** There’s a shortcut, but I’ll do the long way for now:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | 0 | 1 | 2 | 3 |
|  |  |  |  |  |
|  | 0 |  |  |  |
|  | 0 |  |  |  |

Therefore:

Notational, let represent the nth order taylor polynomial of f at a. These have some very nice effects:

***Proposition***: let be a differentiable as necessary and fix some .

1. For any constants
2. If is the derivative of ,
3. If is an anti-derivative of ,

***Proof:*** (Holden 790). Linear is similar to others, the other two rely on previous theorems.

From those proposition, it is also easy to show it is multiplicative (which I won’t show here).

***Example:*** Determine the fourth order Taylor polynomial of the function at the point .

Solution:

Therefore

Example: give the 6th order Taylor Polynomial for at a=0

Solution

### Remainder theorem

This theorem gives you an idea of how close the taylor representation is to the original function. Once you take a Taylor Series, this remainder should be zero, however, taking partial Taylor series will give you a value non-zero remainder that should shrink as you increase the number of terms:

***Theorem***: Let I be an interval with . If is (n+1)-times continuously differentiable on I and is the Taylor remainder for f, then:

## Taylor Series

As mentioned before, Taylor series are Taylor polynomials when the polynomial is infinitely long. With these, we might want to determine the interval of convergence. This is very similar to the other section, so I’ll simply show that not every function has a Taylor series

Solution: Ffirst we need to show that the function is infinetly differentiable (Holden 801).

This shows that not every Taylor series apply uniquely to another function. In the previous example the the Taylor series is . There’s a name for a Taylor function that has a unique function associated;  
Definition: let be a smooth function. We say that f is analytic on I if for every , there is a such hthat

That is , f always locally agrees with its power series. Where I is understood, we just say that f is analytic.

There are some theorems (like Borel’s theorem) that you don’t need to know for MAT137.

## Application of Taylor Series

There are a ton! This ranges from uses in approximation, to solving non-primitive integrals and many more. Unfortunately, we’re just covering 1 in MAT137 2017-18: L’Hopital rule revisted. To use this version of l’hopital, you must be very comfortable with the taylor polynomial expansion of sin, cosine, , log, and functions related to those.

***Definition***: if are two functions and , we say that asymptotically as x approaches a if for every C > 0 there exist s a such that whenever .

The best way to understand is to work through an example.

**Example**: Determine the limit of :

**Solutions**: you know the expansion cos(x). you only need to expand till you get the appropriate term, which in this case is

Notice that as , therefore

Example: determine the limit of

Solution: It might be tempting to develop to

However, the following problem will occur:

Where you can’t solve anymore since you don’t know the constant factors. Instead, do:

Notice that the also factored by 3 ‘x’. you also know that

Therefore